PMM U.S.S.R.,Vol.48,No.1,pp.23-27,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

THE SOLUTION OF BOUNDARY VALUE PROBLEMS FOR THE GENERALIZED CREEP EQUATIONS WITH CONDITIONS IN THE FORM OF INEQUALITIES ON THE BOUNDARY*

A.M. KHLUDNEV

The problem of the existence of solutions of boundary value problems for generalized creep equations with conditions in the form of inequalities on the boundary is considered. In particular, these describe the contact with a rigid body on the boundary. Analogous questions were examined earlier for other creep models /1/ and for an elastic body /2-4/. The method proposed to prove the existence of the solution is constructive and can be used to construct approximate solutions.

Consider the boundary value problem for the generalized creep equations /5, 6/

$$-\sigma_{ij,j}=f_i,\ i=1,\ 2,\ 3 \tag{1}$$

$$\varepsilon_{ij}(u)=c_{ijkl}\sigma_{k,l}+B(t)\,I(s)^{n-1}s_{ij},\ i,\ j=1,\ 2,\ 3 \tag{2}$$

$$\varepsilon_{ij}(u)={}^1/_2(u_{i,j}+u_{j,i}),\quad c_{ijkl}\in L^\infty(\Omega),\quad I(s)=({}^1/_2s_{ij}s_{ij})^{1/2}$$

$$u=0 \text{ on } \Gamma_1\times(0,\ T);\ \sigma_{ij}\nu_j=0 \text{ on } \Gamma_2\times(0,\ T) \tag{3}$$

$$\sigma_\tau=0,\ \sigma_\nu\leqslant 0,\ u_\nu\leqslant 0,\ \sigma_\nu u_\nu=0 \text{ on } \Gamma_3\times(0,\ T) \tag{4}$$

Here σ_{ij} is the stress tensor, s_{ij} is the stress tensor deviator, $u=(u_1,u_2,u_3)$ is the displacement vector, v_j is the derivative with respect to the variable x_j, $B(t)$, n are the creep function and index, $n\geqslant 1$, c_{ijkl} is the elastic-constants tensor that possesses the usual properties of symmetry and positive definiteness, Γ is the smooth boundary of the domain

$$\Omega\subset R^3,\ \Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3,\ \Gamma_i\cap\Gamma_j=\varnothing,\ i\neq j,\ \nu=(\nu_1,\ \nu_2,$$

$\nu_3)$ are components of the exterior normal to Γ, $u_\nu\equiv u\cdot\nu$, σ_ν is the normal component of the vector $\sigma_{ij}\nu_j$, $\sigma_\tau=(\sigma_{1\tau},\ \sigma_{2\tau},\ \sigma_{3\tau})$, $\sigma_{i\tau}=\sigma_{ij}\nu_j-\sigma_\nu\nu_i$. Summation is assumed over the repeated subscripts. Quantities with two subscripts are considered symmetric, $Q=\Omega\times(0,\ T)$, $T>0$, $\sigma=\{\sigma_{ij}\}$, $s=\{s_{ij}\}$. The functions u, σ in problem (1)-(4) are desired.

The following assumption will be used when proving the main result. We shall assume that a solution $\bar\sigma\in C(0,\ T;\ L^{n+1}(\Omega))$ exists for the system of equations (1) which satisfies the second boundary condition of (3) and the first two conditions of (4) (in the distribution sense). It can be shown that a broad class of functions f_i exists for which there is such a solution $\bar\sigma$.

Let W_{Γ_1} be a Sobolev function space

$$W_{\Gamma_1}=\{u=(u_1,\ u_2,\ u_3)\in W^1_{1+1/n}(\Omega)\mid u=0 \text{ on } \Gamma_1\}$$

Theorem. Let $B(t)$ be a continuous function in $[0,\ T]$ such that

$$B(t)\geqslant\beta>0,\ \beta=\text{const},\ \text{mes}\,\Gamma_1>0,\ f_i\in C(0,\ T;\ L^{n+1}(\Omega))$$

and the assumption formulated above relative to the solution $\bar\sigma$ of system (1) is satisfied. Then a solution of problem (1)-(4) exists satisfying the conditions

$$\sigma\in L^\infty(0,\ T;\ L^2(\Omega)),\quad s\in L^\infty(0,\ T;\ L^{n+1}(\Omega)),\ u\in L^\infty(0,\ T;\ W_{\Gamma_1})$$

Proof. The general scheme of the reasoning below is as follows. We first consider the special approximation of problem (1)-(4). The degree of approximation will be characterized by the parameter $\delta>0$. For a discrete analog, in time, of the problem regularized in this manner, we will prove that a solution exists by using the duality method. We then obtain a priori estimates that are uniform in the regularization parameter and in the time spacing. In conclusion, we pass to the limit.

Let N be a positive integer. We set $h=TN^{-1}$ and we introduce the notation

$$A(t)=\{\sigma\in L^{n+1}(\Omega)\mid-\sigma_{ij,j}=f_i(t);\ \sigma_{ij}\nu_j=0 \text{ on } \Gamma_2\}$$
$$A_0=\{\sigma\in L^{n+1}(\Omega)\mid-\sigma_{ij,j}=0;\ \sigma_{ij}\nu_j=0 \text{ on } \Gamma_2\}$$

*Prikl.Matem.Mekhan.,48,1,44-49,1984

Furthermore, δ will be a fixed positive number. For each $m = 0, 1, \ldots, N-1$ we consider the functional

$$H(\sigma) = \frac{1}{2} C(\sigma, \sigma) + \frac{2B(hm)}{n+1} \left[\int_\Omega I(s)^{n+1}\, dx + \delta \int_\Omega I(\sigma)^{n+1}\, dx \right]$$

$$C(\sigma, \alpha) = \langle c_{ijkl}\sigma_{kl}, \alpha_{ij} \rangle$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality between $L^p(\Omega)$ and $L^q(\Omega)$, $p^{-1} + q^{-1} = 1$. Also, let

$$K = \{\sigma \in L^{n+1}(\Omega) \mid \sigma_{ij,j} \in L^{n+1}(\Omega),\ \sigma_\nu \leqslant 0,\ \sigma_\tau = 0 \text{ on } \Gamma_3\}$$

and let $W_{\Gamma_1}^-$ be a closed convex set in W_{Γ_1} defined by the condition $u_\nu \leqslant 0$ on Γ_3. It is known /7/ that we can take

$$\| u \|_{W_{\Gamma_1}} = \left(\int_\Omega |\varepsilon(u)|^{1/\lambda}\, dx \right)^\lambda, \quad \lambda = \frac{n}{n+1}, \quad \varepsilon(u) = \{\varepsilon_{ij}(u)\} \tag{5}$$

as a norm in W_{Γ_1}.

By using this it can be deduced that the set of vectors $e \in L^{1+1/n}(\Omega)$, for which $e_{ij} = \varepsilon_{ij}(u)$, $u \in W_{\Gamma_1}^-$ will be a closed (and convex) subset in $L^{1+1/n}(\Omega)$. Furthermore, we introduce two functionals in $L^{1+1/n}(\Omega)$

$$F(e) = \begin{cases} -\langle f_i(hm), u_i \rangle, & \text{if } e_{ij} = \varepsilon_{ij}(u), u \in W_{\Gamma_1}^- \\ +\infty & \text{otherwise} \end{cases},$$

$$G(e) = \sup_{\sigma \in L^{n+1}(\Omega)} \{\langle \sigma, e \rangle - H(\sigma)\}$$

and we consider the problem of minimizing their sum on $L^{1+1/n}(\Omega)$. By virtue of the definitions of F and G it is equivalent to the following:

$$\inf_{u \in W_{\Gamma_1}^-} [\sup_{\sigma \in L^{n+1}(\Omega)} [\langle \sigma, \varepsilon(u) \rangle - H(\sigma)] - \langle f_i(hm), u_i \rangle\} \tag{6}$$

Let us establish the solvability of the problem obtained in this manner. The equation

$$\langle H'(\sigma), \alpha \rangle - \langle \varepsilon(u), \alpha \rangle = 0, \ \forall\, \alpha \in L^{n+1}(\Omega)$$

holds at the point $\sigma \in L^{n+1}(\Omega)$ where the exact upper boundin (6) is achieved for the given u, Therefore

$$\varepsilon_{ij}(u) = c_{ijkl}\sigma_{kl} + B(hm)[I(s)^{n-1}s_{ij} + \delta I(\sigma)^{n-1}\sigma_{ij}] \tag{7}$$

Substituting the value of $\varepsilon_{ij}(u)$ found into the expression in the square brackets in (6), we find that this expression equals

$$\frac{1}{2} C(\sigma, \sigma) + \frac{2nB(hm)}{n+1} \left[\int_\Omega I(s)^{n+1}\, dx + \delta \int_\Omega I(\sigma)^{n+1}\, dx \right]$$

We again use the fact that the norm in W_{Γ_1} has the form (5). Then we obtain from (7) ($\| \cdot \|_p$ is the norm in $L^p(\Omega)$)

$$\| u \|_{W_{\Gamma_1}} \leqslant c\{ \| \sigma \|_{1+1/n} + \| s \|_{n+1}^n + \delta \| \sigma \|_{n+1}^n \} \tag{8}$$

Here the constant c depends on c_{ijkl}, B, Ω but is independent of u, σ. We therefore have

$$G(\varepsilon(u)) + F(\varepsilon(u)) \geqslant \frac{1}{2} C(\sigma, \sigma) + \frac{2nB(hm)}{n+1} (\| s \|_{n+1}^{n+1} + \delta \| \sigma \|_{n+1}^{n+1}) - \| f(hm) \|_{n+1} \| u \|_{W_{\Gamma_1}}$$

Taking (8) into account, we conclude that $G(\varepsilon(u)) + F(\varepsilon(u)) \to +\infty$ as $\| u \|_{W_{\Gamma_1}} \to +\infty$. Consequently, the existence of a solution in (6) follows from the weak semi-continuity from below of the functional under consideration.

Let $u = u^m \in W_{\Gamma_1}^-$ be the solution of (6). We will prove that the adjoint problem also has a solution. From the definition it follows that $G^*(\sigma) = H(\sigma)$. Moreover

$$F^*(-\sigma) = \sup_{e \in L^{1+1/n}(\Omega)} \{-\langle \sigma, e \rangle - F(e)\} = \sup_{u \in W_{\Gamma_1}^-} \{-\langle \sigma_{ij}, \varepsilon_{ij}(u) \rangle +$$

$$\langle f_i(hm), u_i \rangle\} = \begin{cases} 0, & \text{if } \sigma \in K \cap A(hm) \\ +\infty & \text{otherwise} \end{cases}$$

Let us clarify the last equation. If $\sigma_{ij,j} \neq - f_i\,(hm)$, then obviously $F^*(-\sigma) = +\infty$. Let $\sigma_{ij,j} = - f_i\,(hm)$. We then take a sufficiently smooth function $u \in W_{\Gamma_1}^-$. Writing Green's formula

$$-\langle \sigma_{ij},\ \varepsilon_{ij}(u)\rangle + \langle f_i\,(hm),\ u_i\rangle = -\langle \sigma_\tau,\ u_\tau\rangle_{1/2} - \langle \sigma_\nu,\ u_\nu\rangle_{1/2} \tag{9}$$

we see that the formula for $F^*(-\sigma)$ holds. The brackets $\langle \cdot, \cdot\rangle_{1/2}$ denote the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Since $\sigma, \sigma_{ij,j} \in L^{n+1}(\Omega)$, then σ_τ, σ_ν belong to the space $H^{-1/2}(\Gamma)$ in particular (see /8/) so that (9) has meaning.

Therefore, the adjoint functions are defined. Then the problem adjoint to (6) with respect to the perturbation $\Phi(e, q) = G(e) + F(e - q)$ has the form /9/

$$\inf_{\sigma \in L^{n+1}(\Omega)} \{G^*(\sigma) + F^*(-\sigma)\} = \inf_{\sigma \in K \cap A(hm)} H(\sigma) \tag{10}$$

Because of the convexity and coercivity of the functional $H(\sigma)$, the solution $\sigma = \sigma^m \in K \cap A\,(hm)$ of problem (10) exists.

The relationship /9/

$$G(\varepsilon(u^m)) + G^*(\sigma^m) = \langle \varepsilon(u^m),\ \sigma^m\rangle$$

holds for the solution of the adjoint problems (6) and (10).

Hence it follows that

$$H(\sigma^m) - \langle \sigma^m, \varepsilon(u^m)\rangle \leqslant H(\alpha) - \langle \alpha,\ \varepsilon(u^m)\rangle,\ \forall \alpha \in L^{n+1}(\Omega)$$

This means that the functional $H(\cdot) - \langle \cdot, \varepsilon(u^m)\rangle$ reaches a minimum at the point σ^m, and hence the following equation holds:

$$\langle H'(\sigma^m),\ \alpha\rangle - \langle \varepsilon(u^m),\ \alpha\rangle = 0,\ \forall \alpha \in L^{n+1}(\Omega)$$

from which we conclude that

$$\varepsilon_{ij}(u^m) = c_{ijkl}\sigma_{kl}{}^m + B(hm)\,[I\,(s^m)^{n-1}\,s_{ij}{}^m + \delta\, I\,(\sigma^m)^{n-1}\sigma_{ij}{}^m] \tag{11}$$

We note that u^m, σ^m satisfy the following boundary condition in the weak sense:

$$\sigma_\nu{}^m u_\nu{}^m = 0 \quad \text{on } \Gamma_3 \tag{12}$$

Indeed, the second extremal relationship for the solutions of problems (6) and (10) has the form $F(\varepsilon(u^m)) + F^*(-\sigma^m) = -\langle \varepsilon(u^m), \sigma^m\rangle$, meaning $\langle \varepsilon_{ij}(u^m), \sigma_{ij}^m\rangle = \langle f_i\,(hm), u_i^m\rangle$. Assuming sufficient regularity of the functions recurring here, we obtain

$$\langle \varepsilon_{ij}(u^m), \sigma_{ij}{}^m\rangle = -\langle u_i{}^m, \sigma_{ij,j}^m\rangle + \int_\Gamma \sigma_\nu{}^m u_\nu{}^m\, d\Gamma + \int_\Gamma \sigma_\tau{}^m u_\tau{}^m\, d\Gamma$$

Since $\sigma^m \in K \cap A\,(hm)$, $u^m \in W_{\Gamma_1}^-$, (12) follows.

We obtain arbitrary estimates of the solution. The function σ^m is a solution of (10). Consequently, the inequality

$$\langle H'(\sigma^m),\ \alpha - \sigma^m\rangle \geqslant 0,\ \forall \alpha \in K \cap A\,(hm)$$

holds at the point σ^m.

Substituting the quantity $\bar{\sigma}^m = \bar{\sigma}\,(hm) \in K \cap A\,(hm)$ as α here, we obtain

$$C(\sigma^m, \sigma^m) + 2B(hm)\{\|s^m\|_{n+1}^{n+1} + \delta\,\|\sigma^m\|_{n+1}^{n+1}\} \leqslant C(\sigma^m, \bar{\sigma}^m) +$$
$$B(hm)\{\langle I\,(s^m)^{n-1} s_{ij}{}^m, \bar{\sigma}_{ij}{}^m\rangle + \delta\,\langle I\,(\sigma^m)^{n-1}\sigma_{ij}{}^m, \bar{\sigma}_{ij}{}^m\rangle\}$$

From this inequality the following estimate holds:

$$\max_{0 \leqslant m \leqslant N-1} \{\|\sigma^m\|_2{}^2 + \|s^m\|_{n+1}^{n+1} + \delta\,\|\sigma^m\|_{n+1}^{n+1}\} \leqslant c \tag{13}$$

with a constant c independent of $N, \delta, \delta \leqslant \delta_0$.

Let us now consider the concluding step of the discussion. We pass to the limit as $h \to 0$ and as $\delta \to 0$.

Let σ_h, u_h, B_h be functions, piecewise-constant in t, such that they equal $\sigma^m, u^m, B\,(hm)$, respectively at the points hm. From the estimate (13) and the construction of B_h it follows that a subsequence (notation as before) can be selected for which as $h \to 0$ (δ is fixed)

$$\sigma_h \to \sigma^\delta \ \text{*-weakly in } L^\infty(0, T; L^{n+1}(\Omega)),\ B_h \to B \text{ strongly in } L^\infty(0, T) \tag{14}$$

From (11) we have $\|\varepsilon(u_h)\|_{L^\infty(0,T;L^{1+1/n}(\Omega))} \leqslant c$, so that the sequence u_h is bounded in $L^\infty(0, T; W_{\Gamma_1})$, and consequently, it can be considered that

$$u_h \to u^\delta \text{ *-weakly in } L^\infty(0, T; W_{\Gamma_1}) \tag{15}$$

If $\sigma^m, u^m, B(hm)$ in (11) is replaced, respectively, by σ_h, u_h, B_h, then the relationships obtained will hold over the whole interval $(0, T)$. We pass to the limit in them as $h \to 0$ using (14) and (15). We obtain

$$\varepsilon_{ij}(u^\delta) = c_{ijkl}\sigma_{kl}^\delta + B(t)[I(s^\delta)^{n-1}s_{ij}^\delta + \delta I(\sigma^\delta)^{n-1}\sigma_{ij}^\delta] \tag{16}$$

The justification for the possibility of a passage to the limit in the non-linear expressions is on the basis of the monotonicity of the mapping

$$\sigma \to \{I(s)^{n-1}s_{ij}\},\ \sigma \to \{I(\sigma)^{n-1}\sigma_{ij}\}$$

The proof of the monotonicity can be obtained as follows. We note the equation which has already been used

$$\frac{\partial}{\partial\sigma_{ij}} I(s)^{n+1} = \frac{n+1}{2} I(s)^{n-1}s_{ij} \tag{17}$$

The functional

$$\sigma \to \int_\Omega I(s)^{n+1}\,dx$$

is convex in $L^{n+1}(\Omega)$. A consequence of its convexity will be the monotinicity of the gradient whose components are calculated using (17).

We now pass to the limit as $\delta \to 0$. The estimate (13) is uniform in δ. Consequently, it will hold for the functions σ^δ, s^δ. Moreover, u^δ will be bounded in $L^\infty(0, T; W_{\Gamma_1})$. Selecting a subsequence (notation as before) having the property

$$\sigma^\delta \to \sigma \text{ *- weakly in } L^\infty(0, T; L^2(\Omega)),\ s^\delta \to s \text{ *-weakly in } L^\infty(0, T; L^{n+1}(\Omega))$$
$$u^\delta \to u \text{ *-weakly in } L^\infty(0, T; W_{\Gamma_1}),\ \delta I(\sigma^\delta)^{n-1}\sigma_{ij}^\delta \to 0 \text{ weakly in } L^{1+1/n}(Q)$$

we can pass to the limit as $\delta \to 0$ in (16). The limit functions σ, u yield the solution of (1)–(4). This completes the proof of the theorem.

The function σ will be unique in the theorem obtained.

To prove this it is sufficient to consider relationships (2) which are obtained from (16) after passage to the limit as $\delta \to 0$. These relationships are satisfied in the sense of the identity

$$\int_0^T C(\sigma, \tau)\,dt + \int_0^T \langle u_i, \tau_{ij,j}\rangle\,dt + \int_0^T B(t)\langle I(s)^{n-1}s_{ij}, \tau_{ij}\rangle\,dt = 0 \tag{18}$$

which holds for any sufficiently smooth functions τ satisfying the second boundary condition (3) and the first two conditions (4). In particular, the solution σ can be substituted as τ. If the existence of two different solutions u^1, σ^1 and u^2, σ^2 is assumed, then for $\sigma = \sigma^1 - \sigma^2$ it follows from (18) and (1) that

$$\int_0^T C(\sigma, \sigma)\,dt + \int_0^T B(t)\langle I(s^1)^{n-1}s_{ij}^1 - I(s^2)^{n-1}s_{ij}^2, \sigma_{ij}^1 - \sigma_{ij}^2\rangle\,dt = 0$$

Because of the non-negativity of the expression corresponding to the creep strains, we obtain $\sigma \equiv 0$.

The method of proof proposed for a solution to exist is constructive and can be useful for constructing approximate solutions. Namely, for any δ, h an approximation can be determined for u and σ. For this, two minimization problems must be solved in each layer m:

$$\inf_{e \in L^{1+1/n}(\Omega)} \{G(e) + F(e)\},\quad \inf_{\sigma \in L^{n+1}(\Omega)} \{G^*(\sigma) + F^*(-\sigma)\}$$

The first of these enables the displacement to be determined, and the second the stress. The solution of the initial problem is obtained after passing to the limit as $h \to 0, \delta \to 0$. Introduction of the parameters δ results from the need to regularize the term corresponding to the creep strain. This is explained by the fact that the scheme of investigations assumes an analytically exact description of dual spaces. In this case these spaces are $L^{n+1}(\Omega)$ and $L^{1+1/n}(\Omega)$. However, the stress in the initial problem is an element of the space $\{\sigma \in L^2(\Omega) \mid s \in L^{n+1}(\Omega)\}$ which makes direct construction of the adjoint problems difficult. A result of the adjointness of the above-mentioned problems is the extremal relationships (11). If the passage to the limit is realized therein as $\delta \to 0$, then the equalities obtained in that manner will actually be an extremal relationship for an analog of the initial problem discrete in t. At the same time (2) can also be considered as a certain extremal relationship for the solution of problem (1)—(4). Namely

$$\varepsilon_{ij}(u)=\frac{\partial}{\partial\sigma_{ij}}\left(\frac{1}{2}c_{klpq}\sigma_{kl}\sigma_{pq}+\frac{2}{n+1}B(t)I(s)^{n+1}\right)$$

Note that the Signorini boundary value problems considered in this paper describe the case when the possible area of contact is selected in advance and cannot grow with time (i.e., the greatest possible area of contact is selected), although the presence of contact at any given point is not assumed in advance but is determined only as a result of solving the problem.

REFERENCES

1. ARUTIUNYAN N.Kh. and SHOIKHET B.A., Asymptotic behaviour of the solution of the boundary value problem of the theory of the creep of inhomogeneous ageing bodies with unilateral constraints., Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.3, 1981.
2. FICHERA G., Existence Theorems in Elasticity Theory /Russian translation/, Mir, Moscow, 1974.
3. DUVAUT G. and LIONS J.-L., Les inequations en méchanique et en physique. Paris, Dunod, 1972.
4. KRAVCHUK A.S., On the Hertz problem for linearly and non-linearly elastic bodies of finite dimensions, PMM, Vol.41, No.2, 1977.
5. RABOTNOV Iu.N., Creep of Structure Elements, NAUKA, Moscow, 1966.
6. PISARENKO G.S. and MOZHAROVSKII N.S., Equations and Boundary Value Problems of the Theory of Plasticity and Creep. NAUKOVA DUMKA, Kiev, 1981.
7. MOSOLOV P.P. and MYASNIKOV V.P., Mechanics of Rigidly Plastic Media. NAUKA, Moscow, 1981.
8. LIONS J.-L. and MAGENES E., Problemes aux limites non homogenes et applications. Dunod 1968-1970. (Eng. translation by P. Kenneth, Springer-Grundlehre, 1972-1973).
9. ECKLUND I and TEMAM R., Convex Analysis and Variational Problems /Russian translation/, Mir, Moscow, 1979.
10. KRAVCHUK A.S., On duality in contact problems. PMM, Vol.43, No.5, 1979.

Translated by M.D.F.

PMM U.S.S.R., Vol.48, No.1, pp.27-33, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

NON-LINEAR DEFORMATIONS AND LIMIT EQUILIBRIUM OF THREE-DIMENSIONAL CURVILINEAR RODS*

I.V. SHIRKO

The stress and deformation state of three-dimensional curvilinear rods of circular cross-section is investigated byond the elastic limit. The Kirchhoff-Love hypotheses used. The rod defromations are assumed to be small, but the displacements and the angles of rotation of the central line are arbitrary. The relation between the deformation and the stress states in the plastic region of the material is taken in the form of a linear relation /1/ between the deformation rates and stresses. The coefficients of the equations of this connection are assumed to be specified (for example in the form of a table) by functions of stress components, deformations, time, temperatures, etc. An appropriate selection of these coefficients enables one to describe various models of a solid deformable body.

The method of linearizing the resolving system of equations proposed here enables us to use, for solving specific problems, computational algorithms developed in investigations of geometrically non-linear deformations of elastic rods. It is shown that under specific conditions the elastic kernel, whose cross-section is of elliptic form, degenerates either into a point or a line, and the rod cross-section passes into a purely plastic state. In the purely plastic state the relation between the moments and the force acting over the cross-section is finite, which in the space of generalized force factors (the dimensionless axial force, the twisting and bending moments) are fairly accurately approximated by a sphere. The application

*Prikl.Matem.Mekhan., 48,1,50-57,1984